International Journal of Theoretical Physics, Vol. 20, No. 1, 1981

Strong Gravity and the Yukawa Field

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Received July 15, 1980

A study of strong gravity field coupled to the Yukawa field is carried out for a conformally flat space-time. A quantitative relation between the strong interaction coupling constant $g^2/\hbar c$ and the strong gravity constants ($\Lambda_f \sim 10^{28}$ cm^{-2}, $G_f \sim 6.6\times10^{30}$ C.G.S. units) is obtained giving $g^2/\hbar c \sim 17$, which is of the right order of magnitude. This justifies the contention that strong gravity is relevant for elementary particles (e.g., hadrons).

1. INTRODUCTION

Following the suggestion of Schrödinger (1944) that the charge independence of nuclear forces is analogous to the mass independence of gravitational interactions, some authors have attempted a unified geometrical theory involving both gravitational and scalar meson fields (Murphy, 1977). However, one encounters serious difficulties concerning the strength of the strong (nuclear) interaction in this approach, although the field equations seem to have an acceptable form.

In recent years the role of short-range strong tensor interaction mediated by massive spin-2 (2^+) bosons (so-called strong gravity) (Isham, Salam, and Strathdee, 1971) has been invoked in the study of hadron physics (Sivaram and Sinha, 1973, 1974, 1975, 1976, 1977, 1979; Lord, Sinha, and Sivaram, 1974). This was prompted by the experimental observation of f mesons ($J^P=2^+$) which constitute an $SU(3)$ nonet and interact strongly with hadrons (Lord, Sivaram, and Sinha, 1974; Sinha and Sivaram, 1977). It is found that Einstein-type field equations for such massive spin-2 fields can be formulated. In this the so-called "cosmologi-

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0020-7748/81/0100-0069$03.00/0

cal" term is related with the mass (m_f) of the f meson by the relation $\Lambda_f = (m_f c/\hbar)^2 = 10^{28}$ cm^{-2}. The corresponding field equations, being of Einstein type, constitute a gauge theory of strong spin-2 interaction. The presence of the Λ_f term is equivalent to introducing a mass term in the corresponding Lagrangian and in turn determines the range of strong gravity. Also it connects the strong gravity coupling constant (G_f) with the hadronic density $\rho_h \sim (10^{17}$ g cm^{-3}). We get

$$G_f = \frac{\Lambda_f c^2}{8\pi\rho_h} \approx 10^{38} G_N$$

G_N being the Newtonian constant. One would get the same value for G_f by equating the Compton length of a hadron (say proton) with its "strong" Schwarzschild radius, giving (Sivaram and Sinha, 1979)

$$G_f = \hbar c/2M_p^2 \approx 10^{38} G_N$$

This coupling constant is of the order of the strong interaction of nuclear forces. It is thus felt that a relationship should exist between strong gravity and Yukawa forces. In fact, the unification of scalar meson (Yukawa) fields with strong gravity fields is more natural as they operate within the same range (10^{-14} to 10^{-13} cm) than with the weak (Einstein–Newtonian) gravity (which has infinite range).

In an earlier paper we had studied the problem of strong gravity field coupled to $SO(3)$ gauge field in a conformally flat space (Usha and Sinha, 1979). The solutions of the mass modified field equations of strong gravity gave the potential, which had the expected form $\alpha_f e^{-m_f r}/m_f r$ showing the mediation by a massive (m_f) boson. In the present paper, we investigate the problem of strong gravity coupled to scalar meson (Yukawa) fields. The purpose is to find a quantitative relation between the strong interaction coupling constant ($g^2/\hbar c$) and the constants of strong gravity G_f, Λ_f etc. In Section 2, we formulate the Lagrangian for the coupled system, namely, strong gravity and the Yukawa fields and derive the appropriate field equations. In Section 3 the field equations are written down for a conformally flat space-time. The coupled equations are then solved approximately. It is possible to find a precise relation between the constants of the two fields. Numerical estimates show that $g^2/\hbar c \sim 17$, thus justifying the conjecture that the strong interaction constant is derivable from strong gravity. They are thus intimately related. Finally, we given a short discussion of the relation in Section 4.

2. LAGRANGIAN AND FIELD EQUATIONS

As shown in the previous paper (Usha and Sinha, 1979) we can neglect the effect of weak gravity since such effects appear in lowest order of the quotient $G_N/G_f \sim 10^{-38}$. Accordingly, we are required to write down the action for the strong (f) gravity and the Yukawa field as

$$I = \int L\, d^4x \tag{1}$$

where the Lagrangian has the form

$$L = \frac{(-f)^{1/2}}{2K_f}\left[R(f) - 2\Lambda_f + c_f\left(\varphi_{,\alpha}\varphi_{,\beta}f^{\alpha\beta} - m_\pi^2\varphi^2\right)\right] \tag{2}$$

Here $R(f)$ is the curvature scalar for f-gravity field, Λ_f the "cosmological" constant defined in Section 1, K_f the coupling constant of strong gravity, $K_f = 8\pi G_f/c^4$, and $f_{\mu\nu}$ is the metric tensor of strong gravity. Here φ is the scalar Yukawa field mediated by pions of mass m_π (in units of inverse length) and c_f is a constant $(=1/4\pi)$.

On carrying out variations with respect to $f_{\mu\nu}$ and φ we get the following set of equations:

$$R_{\mu\nu}(f) - \tfrac{1}{2}R(f)f_{\mu\nu} + \Lambda_f f_{\mu\nu} = -K_f c_f\left[\varphi_{,\mu}\varphi_{,\nu} - \tfrac{1}{2}\left(\varphi_{,\alpha}\varphi^{,\alpha} - m_\pi^2\varphi^2\right)f_{\mu\nu}\right] \tag{3}$$

and

$$f^{\mu\nu}\varphi_{;\mu\nu} + m_\pi^2\varphi = 0 \tag{4}$$

where a semicolon represents covariant differentiation.

Equation (4) can also be recast in the form

$$\varphi_{;\alpha}^{;\alpha} + m_\pi^2\varphi = 0 \tag{5}$$

with

$$\varphi_{;\alpha}^{;\alpha} = \frac{1}{(-f)^{1/2}}\frac{\partial}{\partial x_\alpha}\left(f^{\alpha\beta}(-f)^{1/2}\frac{\partial}{\partial x_\beta}\right) \tag{6}$$

representing the covariant d'Alembertian. It may be noted that for flat space-time equation (5) takes the form

$$\frac{d^2\varphi}{dr^2} + \frac{2}{r}\left(\frac{d\varphi}{dr}\right) - m_\pi^2\varphi = 0 \tag{7}$$

which has the well-known Yukawa solution

$$\varphi(r)=\frac{ge^{-m_\pi r}}{r} \tag{8}$$

g being the strong coupling constant (or, in short, strong charge). One is required to solve the set of coupled nonlinear equations given in (3)–(5) to get a relationship between g, G_f, Λ_f, etc.

As done in our previous papers, we choose a simple metric which was found adequate to demonstrate the form of strong gravity potential and its underlying interaction. This is the conformally flat space-time metric given by

$$f^{\mu\nu}=e^{2\lambda}\eta^{\mu\nu} \tag{9}$$

$\eta^{\mu\nu}$ being the flat (Lorentz) metric, and λ is a function of space coordinates

$$f_{\mu\nu}=e^{-2\lambda}\eta_{\mu\nu} \tag{10}$$

Introducing the Einstein-like tensor with the cosmological term, namely,

$$E_{\mu\nu}=R_{\mu\nu}-\tfrac{1}{2}Rf_{\mu\nu}+\Lambda_f f_{\mu\nu} \tag{11}$$

equation (3) can be written in an equivalent form as

$$E_{\mu\nu}=-K_f c_f\left[\varphi_{,\mu}\varphi_{,\nu}-\tfrac{1}{2}f_{\mu\nu}\left(\varphi_{,\alpha}\varphi_{,\beta}f^{\alpha\beta}-m_\pi^2\varphi^2\right)\right] \tag{12}$$

For the metric given in equation (9) the relevant Einstein like tensors $E_{\mu\nu}$ have the forms given below:

$$\begin{aligned}E_{00}&=R_{00}-\tfrac{1}{2}Rf_{00}+\Lambda_f f_{00}\\&=-4\lambda'/r-2\lambda''+\lambda'^2+\Lambda_f e^{-2\lambda}\end{aligned} \tag{13}$$

(where a prime denotes differentiation with respect to r); similarly

$$\begin{aligned}E_{11}&=R_{11}-\tfrac{1}{2}Rf_{11}+\Lambda_f f_{11}\\&=2\lambda''\left(1-\frac{x_1^2}{r^2}\right)-\lambda'^2\left(1+\frac{2x_1^2}{r^2}\right)+\frac{2\lambda'}{r}\left(\frac{x_1^2}{r^2}+1\right)-\Lambda_f e^{-2\lambda}\end{aligned} \tag{14}$$

$$E_{22}=2\lambda''\left(1-\frac{x_2^2}{r^2}\right)-\lambda'^2\left(1+\frac{2x_2^2}{r^2}\right)+\frac{2\lambda'}{r}\left(1+\frac{x_2^2}{r^2}\right)-\Lambda_f e^{-2\lambda} \tag{15}$$

$$E_{33}=2\lambda''\left(1-\frac{x_3^2}{r^2}\right)-\lambda'^2\left(1+\frac{2x_3^2}{r^2}\right)+\frac{2\lambda'}{r}\left(1+\frac{x_3^2}{r^2}\right)-\Lambda_f e^{-2\lambda} \tag{16}$$

where x_1, x_2, x_3 are the space coordinates.

Working out the right-hand side of the equation (12), we get the following four set of equations, namely,

$$E_{00} = -K_f c_f\left[\left(\tfrac{1}{2}(\varphi')^2 + \tfrac{1}{2}e^{-2\lambda}m_\pi^2\varphi^2\right)\right] \tag{17}$$

$$E_{11} = -K_f c_f\left[(\varphi_{,1})^2 - \tfrac{1}{2}\varphi'^2 - \tfrac{1}{2}e^{-2\lambda}m_\pi^2\varphi^2\right] \tag{18}$$

$$E_{22} = -K_f c_f\left[(\varphi_{,2})^2 - \tfrac{1}{2}\varphi'^2 - \tfrac{1}{2}e^{-2\lambda}m_\pi^2\varphi^2\right] \tag{19}$$

$$E_{33} = -K_f c_f\left[(\varphi_{,3})^2 - \tfrac{1}{2}\varphi'^2 - \tfrac{1}{2}e^{-2\lambda}m_\pi^2\varphi^2\right] \tag{20}$$

Assuming that the scalar field φ is time independent and adding the four equations (17)–(20), we get a convenient form of the field equation, namely,

$$2\lambda'' + 4\lambda'/r - 4\lambda'^2 - 2\Lambda_f e^{-2\lambda} = -K_f c_f\left(-e^{-2\lambda}m_\pi^2\varphi^2\right) \tag{21}$$

Similarly, working out the covariant d'Alembertian for the conformally flat metric, the explicit form of equation (5) turns out to be

$$e^{2\lambda}(\varphi'' + 2\varphi'/r - 2\lambda'\varphi') - m_\pi^2\varphi = 0 \tag{22}$$

which in the limit of flat space-time reduces to equation (7) having the Yukawa solution given by equation (8).

3. SOLUTIONS OF EQUATIONS

Before taking up the solution of the complete equation given in (21) and (22), it is expedient to first consider the uncoupled situation. In the absence of the Yukawa source the strong gravity field equation for conformally flat space-time has the form

$$2\lambda'' + 4\lambda'/r - 4\lambda'^2 - 2\Lambda_f e^{-2\lambda} = 0 \tag{23}$$

[cf. equation (21), where the right-hand side is put equal to zero]. On making the substitution

$$F = re^{-2\lambda} \tag{24}$$

the above equation reduces to

$$F'' + 2\Lambda_f F^2/r = 0 \tag{25}$$

which is still nonlinear. Now in the absence of the cosmological term Λ_f the above reduces to

$$F''=0 \tag{26}$$

This has the general solution

$$F=ar+b \tag{27}$$

where a and b are constants. We choose $a=1$, giving

$$e^{-2\lambda}=1+b/r \tag{28}$$

Thus in the asymptotic limit $r\to\infty$, $e^{-2\lambda}\to 1$, giving the correct flat space-time limit. Thus, for large r, $e^{-2\lambda}\to 1$ and $F\to r$. Taking this limiting behavior for F in the nonlinear part of equation (25), we get

$$F''=-2\Lambda_f r \tag{29}$$

which has a solution

$$F=r-\Lambda_f r^3/3+b \tag{30}$$

or

$$e^{-2\lambda}=1+b/r-\Lambda_f r^2/3 \tag{31}$$

and with $b/r=-2G_f M/c^2 r$, is like the Schwarzschild solution. The additional term $\Lambda_f r^2/3$, is similar to the de Sitter form that occurs in other spherically symmetric solutions for nonzero cosmological terms (Adler, Bazin, and Schiffer, 1965).

As a first approximation, we attempt to get a relation between g, G_f, and Λ_f of equation (21) by using the solution given by equations (30) and (31), and the Yukawa solution. In the present context, space-time is taken to be flat beyond the range of strong gravity which is of the order of 10^{-14} cm. Equation (21) takes the form (with $c_f=1/4\pi$),

$$\begin{aligned} F'' &= -2\Lambda_f \frac{F^2}{r} - \frac{K_f}{4\pi}\frac{F^2}{r} m_\pi^2 g^2 \frac{e^{-2m_\pi r}}{r^2} \\ &\approx -2\Lambda_f r - \frac{K_f}{4\pi} g^2 m_\pi^2 \frac{e^{-2m_\pi r}}{r} \end{aligned} \tag{32}$$

where in all the nonlinear terms on the right-hand side we put the asymptotic (flat space-time) limit $F \to r$, $e^{-2\lambda} \to 1$. The solution of equation (32) is then obtained by expanding the exponential and integrating. With a suitable choice of constants of integration, we get

$$F = r - \frac{2G_f M_p}{c^2} - \frac{\Lambda_f r^3}{3} - \frac{2G_f g^2 m_\pi^2}{c^4}\left(r \ln r - m_\pi r^2 + \frac{1}{2m_\pi}\right) \tag{33}$$

where M_p is a typical hadron mass (e.g., proton). This in turn gives

$$e^{-2\lambda} = 1 - \frac{2G_f M_p}{c^2 r} - \frac{\Lambda_f r^2}{3} - \frac{2G_f g^2}{c^4} m_\pi^2 \left(\ln r - m_\pi r + \frac{1}{2m_\pi r}\right) \tag{34}$$

Although the last term is obtained by an approximation method (Migdal, 1977), it gives the right answer, in the sense that the f given by equation (33) is a solution of equation (32) within the approximate regime considered. The gravitational potential is given by (Adler, Bazin, and Schiffer, 1965)

$$V = \frac{c^2}{2}(f_{00} - 1) = \frac{c^2}{2}(e^{-2\lambda} - 1) \tag{35}$$

We now use the ansatz that for stable elementary particles, (e.g., proton) the strong gravitational force is zero at the radius of the stable hadron (Ross, 1972; Perng 1978), i.e.,

$$\left.\frac{\partial V}{\partial r}\right|_{r=r_p} = 0$$

thus we get

$$\frac{2G_f M_p}{c^2 r_p^2} - \frac{2\Lambda_f r_p}{3} + \frac{G_f g^2 m_\pi}{c^4} \frac{e^{-2m_\pi r_p}}{r_p^2} = 0 \tag{36}$$

From this relation the dimensionless strong coupling constant $g^2/\hbar c$ is easily obtained as

$$\frac{g^2}{\hbar c} = \frac{\exp(2m_\pi r_p) c^3 r_p}{m_\pi G_f \hbar} \left[\frac{2\Lambda_f r_p^2}{3} - \frac{2G_f M_p}{c^2 r_p}\right] \tag{37}$$

Using the numerical values of various quantities, namely,

$$\Lambda_f \sim 10^{28}\ \text{cm}^{-2},\ G_f \sim 6.6\times 10^{30}\ \text{(C.G.S. units)}$$

$$M_p\ \text{(proton)} = 1.67\times 10^{-24}\ \text{g},\ c = 2.998\times 10^{10}\ \text{cm/sec}$$

$$m_\pi (= m_\pi c/\hbar) = 0.68\times 10^{13}\ \text{cm}^{-1},\ \hbar = 1.05\times 10^{-27}\ \text{erg sec}$$

and

$$r_p = 2.00\times 10^{-14}\ \text{cm}, \qquad \exp(2m_\pi r_p) \sim 1$$

[The value of r_p is of the same order as the proton Compton length and its Schwarzschild radius for strong gravity.] We get

$$g^2/\hbar c = 17$$

This compares very favorably with the observed value 14.50.

4. CONCLUDING REMARKS

In the foregoing sections, we have presented a calculation for strong gravity coupled to Yukawa field. The purpose was to get a relationship between the strong (nuclear) interaction coupling constant ($g^2/\hbar c$) and the parameters of the strong gravity field, namely, Λ_f, G_f, etc. Einstein-type equations were used for strong gravity as suggested in some of our earlier work. We have chosen a conformally flat space-time for the strong gravity calculation, so that there is only one potential (effectively the equation for λ) unlike the ten of Einstein's weak gravity. As the strong gravity (strictly strong spin-2 interaction) is relevant to hadron physics, we did not wish to enter into complicated calculation at this stage of the development of the field. However, it should be noted that the solution obtained has all the essential elements, i.e., Schwarzschild, de Sitter, and the right form of the strong charge solution. Further, we obtain the value $g^2/\hbar c \sim 17$, which is indeed encouraging and justifies the premises.

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